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# Xiangqi and Combinatorial Game Theory

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**Abstract**

We explore whether combinatorial game theory (CGT) is suitable for analyzing endgame positions in Xiangqi (Chinese Chess). We discover some of the game values that can also be found in the analysis of International Chess, but we also experience the limitations of CGT when applied to a loopy and non-separable game like Xiangqi.

## 1 Introduction

Most people believe that Xiangqi (Chinese Chess) and Shogi are variants of Chess (by Chess we always mean International Chess) which was invented by the Indians in the $6^{th}$ Century. But this is probably not true. Rather, Xiangqi and Backgammon evolved from an ancient Chinese game called Liubo that was invented some 3,500 years ago [14, pp. 3–11]. Like Liubo, the modern Xiangqi consists of one king, five pawns, and several higher order pieces, and the game is decided by capturing the opponent's king. Unlike in Liubo, the moves of modern Xiangqi are not determined by the roll of dice. The dice part of the game eventually evolved into another famous game, Backgammon. Based on this revelation, Xiangqi predated both Shogi and Chess, and the latter are obviously variants of the former instead.

Since Xiangqi and Chess are purely strategic games without random moves, it should, at least in principle, be possible to decide whether there is a winning strategy for the first or the second player, or whether the game always ends with a draw (assuming perfect play). Such games are called *combinatorial games*, and *combinatorial game theory (CGT)* is a branch of mathematics devoted to their analysis [3]. A rich theory on how to evaluate game positions has been developed in recent years, and it has been successfully applied to analyze certain endgame positions of Chess [8] and Go [4], for example.

Unfortunately, CGT cannot directly be applied to Xiangqi (or Chess). In Section 2 we will therefore make a few simplifying assumptions about the game to make it better accessible to CGT. These assumptions will of course affect the actual value of a game position but we will still be able to correctly identify the winner.

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In Chess, pawn endgames can be easily analyzed if there are no higher order pieces are on the board [8]. The situation in Xiangqi is not so easy, unfortunately. Pure pawn endgames are rare because higher order pieces usually survive until the end, leading to loopy games with more complex values.

Another problem in strategic games is to find a winning move (or at least a good move) for a given game position. Since the search space is usually astronomically large, finding the best move by exhaustive search is not a good idea. Actually, most interesting games are PSPACE- or EXPTIME-complete (would a game with an easily computable winning strategy not be quite boring?). We believe that Xiangqi, like Chess [10] and Go [16], is EXPTIME-complete. There exist many endgame rules forbidding the perpetual chase of some opponent's piece [1, 13], and experience with other board games (like Go with the Ko-rule) shows that such rules usually make a game polynomially intractable.

In Section 2 we give a short introduction of CGT. And we propose a few restrictions to the movements of the pieces to make Xiangqi better accessible to CGT. In Section 3 we analyze pawn endgame positions that show that some of the 'usual' game values of CGT also appear as value of Xiangqi positions. In particular, we construct game positions with integral, fractional, and infinitesimal values. In Section 4 we discuss game values of realistic positions, without the restrictions of Section 3. In particular, we construct game positions of value `over`, `tis` and `tisn`, `on` and `off`, `dud`, and `nimbers`. We conclude with a few general remarks in Section 5.

# 2 Basic Definitions

We first give a short introduction of CGT. Then we propose a few restrictions to piece movements in Xiangqi to make the game better suited for analysis by CGT.

## 2.1 Combinatorial Game Theory

A *combinatorial game* is a two-player perfect-information game (both players have access to the same information) without chance moves (no dice) [2, 6, 9, 11] (see also a recent survey by Demaine [7]). The number of *positions* is usually finite, and there is a particular *starting position*. The players are usually called *Left* and *Right*. In Xiangqi, we use *Red* for Left, and *Black* for Right. The players move alternately according to clearly defined *rules*, usually with Left making the first move. The positions that can be reached by some move of Left (Right) are called Left's (Right's) *options*.

A *game* can be represented by its game graph, where nodes are the game positions and directed arcs are the options. A *play* is a path in this graph from the starting node to some *terminal node*, or an infinite path (a *draw*). A game with draws is called *loopy*. In *normal play*, the first player unable to move *loses* the play and his opponent *wins* (in *misère play* the first player unable to move wins). We say a player *wins a game* if he has a winning strategy, i.e., he can always force the play to end in a winning position. A *draw* is a position from which neither player can force a win, and each player has a non-losing next move. Some games (like Chess) also know *ties*, i.e., positions where the play ends but neither player wins.

Typical examples of combinatorial games are Chess, Nine-Men Morris, Checkers, and Go. Games where the players do not have the same moves available from the same game position are called *partizan games*, otherwise they are *impartial games*. Note that in the game graph of partizan games we must distinguish between edges corresponding to Left's and Right's options. In impartial games, each player can use any edge of the game graph.

A combinatorial game without draws or ties can have four different outcomes (assuming perfect play): Left/Right player wins (whether he starts or not), or first/second player to move wins. The ultimate goal of *combinatorial game theory (CGT)* is to classify all combinatorial games into these four categories.

*Algorithmic combinatorial game theory (ACGT)* on the other hand studies the complexity of computing winning moves. Even if it is known which player wins, it can be very difficult to find a winning move. It is known that in Hex the first player has a winning strategy but this strategy still has to be discovered [5].

To better understand the structure of games Conway introduced the *surreal numbers* [2, 6, 12], generalizing the real and ordinal number system. They are given by a simple recursive definition [6, pp. 4]:

> *If $L, R$ are any two sets of numbers, and no member of $L$ is $\geq$ any member of $R$, then there is a number $\{L \mid R\}$. All numbers are constructed in this way.*

The simplest number is $\{\ \mid\ \}$, i.e., $L = R = \emptyset$. This number is called 0. We can now iteratively construct numbers $\{0 \mid\ \} = 1$, $\{0, 1 \mid\ \} = 2, \ldots, \{\ \mid 0\} = -1, \ldots$, $\{0 \mid 1\} = \frac{1}{2}, \ldots$, etc. With addition and multiplication defined in a natural way, the surreal numbers form a field which includes both the real numbers and the ordinal numbers. In particular, the surreal numbers are totally ordered.

It turns out that the surreal numbers are actually a subclass of the class of combinatorial games. A game position $G$ is completely characterized by Left's options $L$ and Right's options $R$, and we write $G = \{L \mid R\}$. Note that there is no constraint on $L$ and $R$ as in the recursive definition of the surreal numbers. For example, a game $\{x \mid y\}$ where $x$ and $y$ are numbers and $x \geq y$ is called a *switch*. Some of the algebraic structure of the surreal numbers, such as addition and the order relation, carries over to games, but games are no longer totally ordered.

If two non-identical games have the same value ($\{\ \mid\ \} = 0 = \{-1 \mid 1\}$, for example) then they also have the same outcome (assuming perfect play). In particular, in any game of value 0 the second player to move wins, because in the *endgame* $\{\ \mid\ \}$ neither player has an option and the first player to move immediately loses. In all positive games (i.e., of value larger than 0), like $\{0 \mid\ \} = 1$ for example, Left wins. Intuitively, in this case the value of the game characterizes the number of spare moves Left could make after winning the game.

Symmetrically, Right wins in all negative games. And if a game $G$ is *fuzzy*, or *confused with* 0 (i.e., neither $G < 0$, $G = 0$, nor $G > 0$), then the first player to move wins. In this case, we write $G \parallel 0$. For example, $\{0 \mid 0\} \parallel 0$ because the first player can move to game 0 where the second player loses (because he is now the first player in this game of value 0). This game is called $\star$. Note that games can also be confused with other numbers than 0, but then they are not fuzzy, i.e., first player wins. For example, $\{3 \mid 2\}$ is confused with 2 but nonetheless positive, i.e., Left wins this game.

Other interesting games are the infinitesimals $\{0 \mid \star\} = \uparrow$ ('up'; positive, but smaller than any positive number), $\{\star \mid 0\} = \downarrow$ ('down'), $\uparrow + \uparrow = \Uparrow$ ('double-up'), $\{0 \mid \downarrow + \star\} = \uparrow^2$ ('up-second'; positive, but somewhat smaller than $\uparrow$), $\{\uparrow + \star \mid 0\} = \downarrow_2$ ('down-second'), $\uparrow + \uparrow^2 = \uparrow 2$ ('up-two'), and $\{0 \mid \{0 \mid -2\}\} = +_2$ ('tiny-two'; positive, but much smaller than $\uparrow$).

## 2.2 Assumptions on the Rules of Xiangqi

In Xiangqi (Chess) we will always call the two players Red (White in Chess) and Black instead of Left and Right, respectively. All games will be seen from Red's perspective. We will not use the traditional way of writing down Xiangqi moves. Instead, we will use the more intuitive Chess notation (sometimes also called algebraic notation). Note that contrary to Chess in Xiangqi the rows of the board are numbered from top to bottom.

When we try to apply CGT to analyze Xiangqi positions we face several difficulties. CGT assumes that a game always has a winner, there is no room for draws. Therefore, Chess and its variants are strictly speaking not combinatorial games. But Moews [15] showed that it is possible to extend CGT to Go, also a loopy game. Since in Xiangqi no player seems to have a real advantage from choosing repeating moves (except when any other move would be losing and the repeating move would force the opponent to repeat his answer; in this situation, the game would be a draw) we restrict them upfront.

**Assumption 1:** *We assume that kings do not move.*

With this assumption, the position in Fig. 1(a) has value $\star$ (see Subsection 3.2), i.e., it is a first player win. But without the assumption the second player's best option would be to perpetually move its king, eventually forcing the first player to do the same, resulting in a draw. Berlekamp et al. [2, pp. 317] call this situation a *dud*.

Assumption 1 is actually equivalent to say that the subgame played by the two kings on their respective $3 \times 3$ roaming areas has value zero. Since the kings could move forever, resulting in a draw, this is not quite right. But it is close enough since we could augment CGT to include draws by arbitrarily treating a draw as a second player win (at least, it is a position where the first player cannot win).

We put a similar restriction on pawns. Since pawns can move horizontally after crossing the river in the middle of the board they are another potential source of loopy games.

**Assumption 2:** *We assume that pawns can only move horizontally if there is a possibility to capture an opponent's piece.*

In particular, after the opponent has lost all his pieces the pawns can only move forward. This allows the pawns to move horizontally, but only in one direction until they hit the board boundary. With this assumption we will usually underestimate the value of a game position because we deprive a player of additional spare moves, but we will still usually predict the right winner.

Elkies [8] observed that another drawback of Chess is the rather small board size, compared to the Go board. In particular, most pieces can have a far-reaching impact. This is also true for Xiangqi. As a consequence, it is very difficult to subdivide a game into several independent subgames, a key step in CGT analysis. Although the Xiangqi board is slightly bigger ($9 \times 10$) than the Chess board ($8 \times 8$) it has only five pawns besides the pairs of higher order pieces (rook, guardian, cannon, and elephant). The rules of Xiangqi [1, 13] are more complex than in Chess and players rely more on higher order pieces rather than pawns. To make CGT work we restrict the arbitrary movements of the pieces.

**Assumption 3:** *We assume we can analyse subgames on different parts of the board independently.*

We usually consider subgames on a single file or on a collection of files. We denote by $val(x)$ the value of the subgame on file `x`, and by $val(x - y)$ the subgame on files `x`, `x+1`, ..., `y`. The assumption that pieces in one subgame cannot interact with the pieces in another subgame is clearly a strong restriction, and it might easily result in a wrong evaluation of a game position.

In Chess pawn endgames, a subgame usually ends with the pawns blocking each other. This is not possible in Xiangqi. Therefore, we only analyze subgames until the first piece is captured.

**Assumption 4:** *We assume that a subgame ends either when some piece is captured or when no piece can move.*

Conceptually, after a piece has been captured all other pieces are considered immobile, so we consider this to be a subgame of value 0 (i.e., a second player win; note that the second player is the player who just won the subgame by capturing a piece). In Subsection 3.2 we will discuss some implications of Assumption 4. We note that Elkies implicitly introduced a similar assumption when he analyzed Chess positions [8]. He lets a game end when one player could be forced to move a piece in a trébuchet (mutual Zugzwang) position, thus eventually losing the game.

# 3 Pawn Endgames

Compared to Chess, pawns in Xiangqi seem to be weaker because they are never elevated to higher order pieces. But their movement becomes more complicated when they cross the river, greatly increasing their power. Fortunately (or unfortunately), pawns cannot be blocked by an opponent's pawn on the same file as capture is made on the same file. Thus, pawns will usually survive longer than in Chess.

## 3.1 Integral and Fractional Values

In Fig. 1(b), the subgame on file `i` has value $val(i) = 0$ because the first player to move immediately loses his pawn. And we can create positions with arbitrary

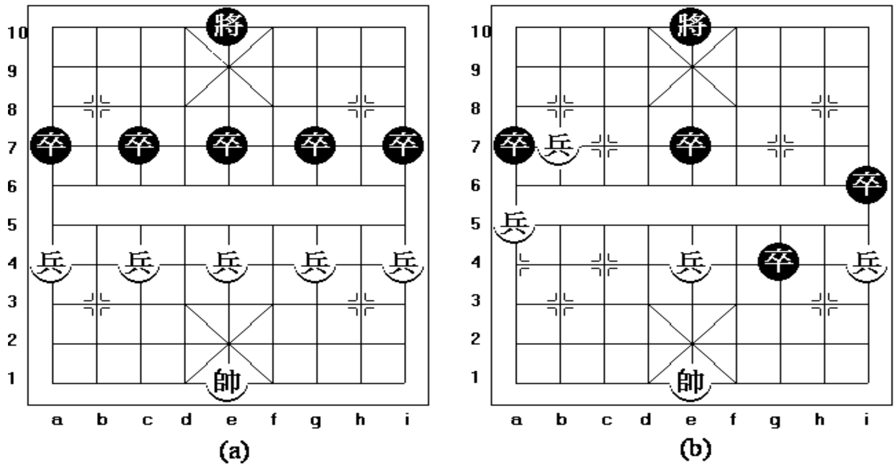


Figure 1: Two fuzzy games: (a) value $\star$; (b) value $\{\uparrow \mid 0, \star\}$.

integer values on a single file by placing one Red pawn on the black half of the board and one Black pawn on the red half of the board. These pawns cannot interact, they can only move foward until they reach the baseline. The value of this subgame is then the difference between the number of Red's moves and the number of Black's moves. Having subgames of integer value on single files, we can easily create subgames of fractional value ($\frac{1}{2} = \{0 \mid 1\}$, for example).

It seems impossible to create more complex positions of integral or fractional value because, by Assumption 4, a subgame ends when some piece is captured. As a consequence, there are no spare moves for the winner after capturing a piece (and the number of spare moves is intuitively the value of a winning position).

## 3.2 Infinitesimal Numbers

We can easily construct pawn subgames of infinitesimal value. In Fig. 1(a), the pawns and kings are at their initial positions.

File `a` is win for the first player because $val(a) = \{0 \mid 0\} = \star$. Analogously, $val(c) = val(e) = val(g) = val(h) = \star$. The full board in Fig. 1(a) has value $\star + \star + \star + \star + \star = \star$, pertaining the game to be a win for the first player to move.

In Fig. 1(b), we first consider the sub-game of files `a` and `b`. Red has two options, 1. `a5-a6` and 1. `b7xa7`. The former move will lead to a position of value $\star$, and the latter move will lead to a position of value 0. On the other hand, Black has only one option, 1. ... `a7-a6`, leading to a position of value $\star$. Thus, $val(a-b) = \{0, \star \mid \star\}$. Since Red's option of $\star$ provides no real advantage (it is a reversible move [6, pp. 110] that could be bypassed, see [3, pp. 62–66]), we see that $val(a-b) = \{0 \mid \star\} = \uparrow$. So the subgame on files `a-b` is a winning position for Red, although with infinitesimally small advantage (i.e., Red does not have a full spare move).

What would be the value of the subgame without Assumption 4? The red pawn on `b7` could move to the left to capture the black pawn on `a7`, but afterwards could only move forward on the `a` file (3 more moves). Also, if Black would first move is pawn to `a6` then the red pawn could only move forward on the `b` file and never enter the `a` file. Thus, we would obtain $val(a-b) = \{7 \mid \{7 \mid -1\}\} = 7 + +_8$, a value only infinitesimally larger than 7. So the difference between the two game values, with or without Assumption 4, is approximately the seven extra moves Red gains after the black pawn has been captured, but, somehow surprisingly, the difference is not exactly 7.

File `e` has value $val(e) = \star$. Thus, $val(a-b, e) = \uparrow\star$. Indeed, the two subgames on files `a-b` and `e` can be considered to be independent because the red pawn on `b7` could not reach the black pawn on `e7` before it had to move forward to `b6`.

As we have seen before, file `i` alone would have value $val(i) = 0$. But we must also consider the black pawn on `g4`, so we consider the subgame consisting of the files `g` to `h`. Red has only one option, 1. `i4-i5`, leading to a position of value $\star$. On the other hand, Black has two options, 1. ... `i6-i5` and 1. ... `g4-h4`. The former move also leads to a position of value $\star$. The latter move leads to a position symmetrical to the position on the `a-b` file, so it has value $-val(a-b) = \downarrow$. Thus, $val(g-i) = \{\star \mid \star, \downarrow\} = \{\star \mid \downarrow\} = \downarrow 2$. So this subgame is a winning position for Black. But what is the value of the full board?

Intuitively, we would expect a small advantage for Black because the position is nearly symmetrical, except for an additional spare move for Black with the pawn on `g4`. Indeed, we have $val(a-b, g-i) = \{\uparrow\star \mid 0\}$, an infinitesimally small negative value. For the value of the full board we obtain $\uparrow + \star + \downarrow 2 = \{\uparrow \mid 0, \star\}$ which is fuzzy, i.e., the board is a first player win. Red could start with 1. `i4-i5`, leaving Black only the options 1. ... `i6xi5` (where Red would win with 2. `e4-e5`), 1. ... `e7-e6` (where Red would win with 2. `i5xi6`; note that this move would end the subgame on files `g-i`), and 1. ... `a7-a6` (where Red would win with 2. `i5xi6`). And similarly, Black could win by first moving 1. ... `a7-a6`.

# 4 Real Endgames

To analyze pure pawn endgames we introduced restrictions on the pawn movements (Assumption 2) to prevent loopy games. For higher order pieces we cannot easily do the same trick. Also, Assumption 4 does not seem to be useful anymore (we want to be able to capture and capture back).

## 4.1 King Plus Pawn Against King

An important endgame position is Red with king and one pawn against Black only with the king (see Fig. 2(a)). This position is a win for Red. A winning move sequence is, for example, 1. `f7-f8` `Ke10-e9`, 2. `Kf1-f2` `Ke9-e10`, 3. `f8-f9` `Ke10-d10` 4. `f9-e9` — stalemate.

What is this game's value? Red could straightforward attack the Black king, but it could also delay the attack by arbitrary king or pawn moves. Black can only sit there and wait for Red's attack. This is an example of a game called `over` [2, pp. 321].

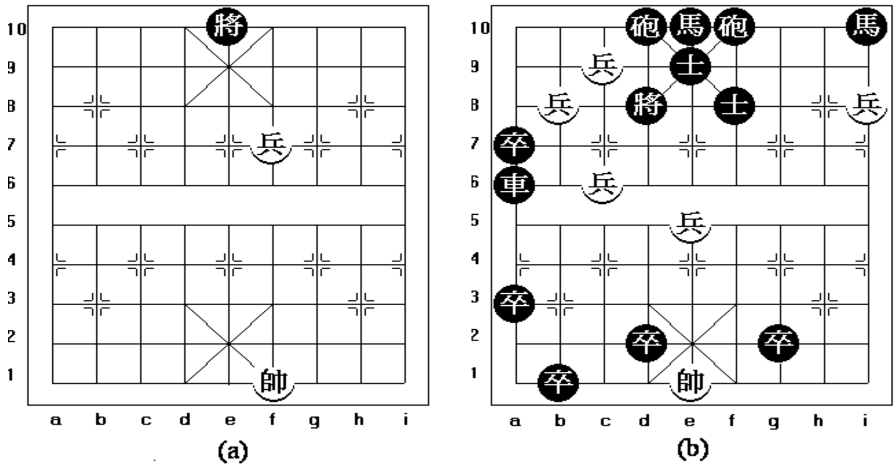


Figure 2: (a) King plus pawn against king; (b) A classical endgame.

Adding a Black pawn in Fig. 2(a) would make the game infinite as neither of the opponents could force a win. Such a game is called a `dud` [2, pp. 317]. Actually, `dud= on+off` (see also Subsection 4.5).

## 4.2 A classical endgame

Analyzing endgames can be quite tricky (for humans) if also higher order pieces are involved. Fig. 2(b) shows a classical endgame [13, pp. 77]. Red seems to be overpowered and outnumbered by Black. We cannot apply CGT to analyze this game. Remarkably, Red has a winning startegy if he can move first. All he has to do is threaten the black king: 1. `b8-c8 Kd8-e8`, 2. `e5-e6 Ge9-d8` (to avoid a mate from 3. `e6-e7`), 3. `c9-d9 Gf8-e9` (to avoid another mate), 4. `e6-e7 Ke8-f8`, 5. `e7-e8 Kf8-f9` (in order not to expose the king directly), 6. `e8xe9 Gd8xe9`, 7. `d9-e8 Kf9-f8`, 8. `c8-d8 Ni10-g9` (to avoid another mate from `d8-c8`, Black moves his knight), 9. `i8-h8` (another threat on `g8`) `d2-d1`, 10. `Ke1-e2 g2-f3`, 11. `Ke2-e3 Cf10-g10`, 12. `d8-e8 Ng9xe8`, 13. `h8-g8` — the black king is trapped.

## 4.3 King Plus Rook Against King

Fig. 3(a) shows a game where Red has the king and a rook, while Black only has the king. Red can win this game by occupying the central file `e` with the rook and the king, followed by check-mating the black king either on file `d` or `f`. But again, Red can delay the winning moves, so this game has also value `over`.

## 4.4 Tis and Tisn

Consider the following game. Red has a pawn on `h6`, and Black has a pawn on `g7`. Red does not want not move `h6-h7` because then the black pawn could

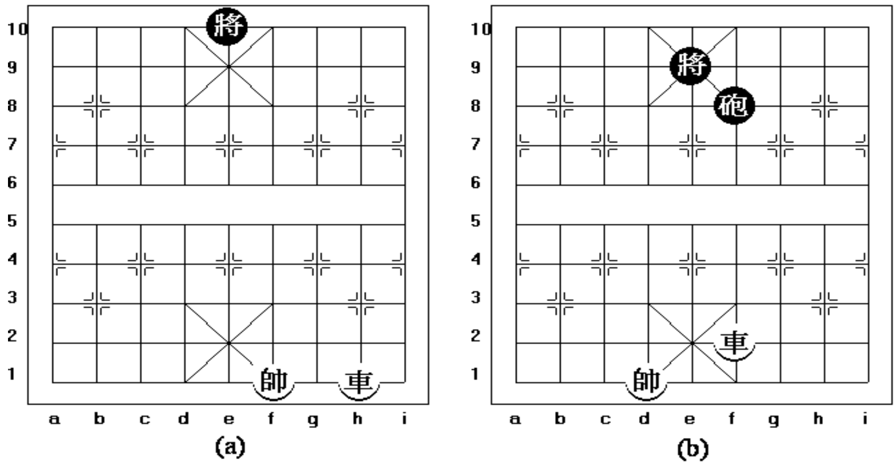


Figure 3: (a) King plus rook against king; (b) `tis` and `tisn`.

move forward and the game would end as a draw. Red does also not want to move `h6-g6` because it would lose immediately. Similarly, Black does not want to move `g7-g6`.

This is an example of a game called *tis* and *tisn* [2, pp. 322]. If Black moves a pawn first he loses the game. And if Red moves a pawn first he either loses the game or it ends in a draw. Thus, both sides always move their kings, and we have again an infinite game.

If the black pawn started at `g8` instead of `g7` then the situation would be better for Red. Red's first move 1. `h6-g6` would result in a subgame of value $val(g-h)=0$ (which is better than `tis`), while Black's only option 1. ... `g8-g7` would lead to the previous game. We therefore write `tis` $= 1 +$ `tisn`.

Fig. 3(b) is another example of `tis` and `tisn`. Assume Red just moved the rook to `f2` to threaten the black cannon. Black's best answer is 1. ... `Ke9-f9`. If Red captures the rook the game will end as a draw: 2. `Rf2xf8 Kf9xf8`, and the two Kings have infinite moves.

### 4.5 On and Off

The simplest loopy games are `on` and `off` [2, pp. 316–320]. These are two independent subgames, where in `on` Red can move forever undisturbed by Black's moves, whereas in `off` Black can move forever undisturbed by Red's moves. These games could of course be constructed with pawns that move horizontally forever after crossing the river. But we can also employ guardians to get these positions. Their movement is restricted to the castle so they cannot interact with the opponent's king or guardians. But they can move as often as they like. Thus, a subgame where Red (Black) has one guardian and the king is an `on` (`off`). And the sum of these two games is a `dud`, i.e., an infinite game where no player can win or lose.

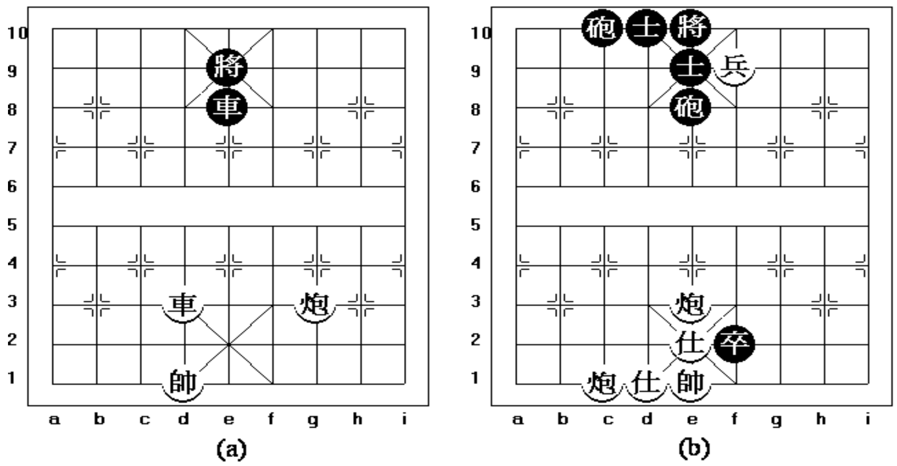


Figure 4: (a) A strange game; (b) A nimber.

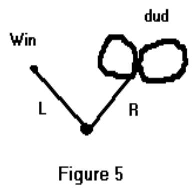


Figure 5: Game graph of a strange game.

## 4.6 A Strange Game

Fig. 4(a) (see [14, pp. 46] for a discussion of this positon) is an example of a game not discussed in [2]. If Red moves first he can win by 1. `Cg3-e3`. However, if Black has the first move he can start with 1. ... `Re8-e2`. Then Red can never occupy the `e` file and the game ends as a draw, i.e., we have a `dud`. The game graph of this game is shown in Fig. 5.

## 4.7 Nimbers

We can also construct Xiangqi positions that are equivalent to positions of the game Nim. These game values play an important role in CGT and they are called *nimbers* [3, pp. 41]. Nim is played on heaps of pebbles. Both Left and Right play alternatingly taking away some pebbles from one of the heaps. The first player not able to move loses. In Poker-Nim [3, pp. 53], they can also put back some of the pebbles they have taken away earlier in the game. It turns out that Nim and Poker-Nim are actually equivalent.

Now consider the game in Fig. 4(b). Both Red and Black have two cannons and two guardians. None of the pieces are able to move (without immediately losing the game) except the cannons, and even they can only move vertically on their respective file. Thus, this position is equivalent to a Nim position. The files with the cannons correspond to heaps, and the distance between two cannons corresponds to the number of pebbles in the heap. For example, file `c` corresponds to a heap of size 8, and file `e` corresponds to a heap of size 4. It is not difficult to see that this particular position is a first player win.

## 5 Conclusions

We have applied CGT to the analysis of certain Xiangqi positions. In many cases, we were only able to carry out a meaningful analysis if we assumed certain restrictions on the movement of pieces on the board. Without these restrictions, most Xiangqi positions seem to be beyond the power of CGT (we only see that the games are loopy and infinite). The greatest challenge is to incorporate sums of non-independent subgames into CGT.

## 6 Acknowledgements

We are grateful for the existence of David Wolfe's Gamesman's Toolkit [17], a very useful tool for computing values of game positions.

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