COMBINATORIAL GAMES: MODULAR N-QUEEN

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Abstract. The classical n-queen problem has been used as a benchmark for goal searching in a solution space by the Artificial Intelligence community. The original problem was confined to 8 queens on a regular chessboard. Over the years, it has transformed into many forms. In this paper, we will talk about yet another variation of the n-queen problem, the *Modular n-queen Problem*, and ask if the readers are able to answer the open problems.

Introduction. Chess was invented by the Indians in the 6th Century. On the other hand Xiangqi and Backgammon evolved from an ancient Chinese game called Liubo that was invented some 3,500 years ago [Lin 1991].

Chess and its variants, are purely strategic games without any random moves, it should, at least in principle, be possible to decide whether there is a winning strategy for the first or the second player, or whether the game always ends with a draw (assuming perfect play). Such games are called *combinatorial games*, and *combinatorial game theory* (CGT) is a branch of mathematics devoted to their analysis [Berlekamp et al. 1982]. A rich theory on how to evaluate game positions has been developed in recent years, and it has been successfully applied to analyze certain endgame positions of Chess [Elkies 1996] and Go [Berlekamp et al. 1994], for example. Unfortunately, CGT cannot directly be applied to Chess and Xiangqi, due to the fact that draws do not qualify as a game in CGT [Berlekamp et al. 1982].

What is the n-queen problem? Consider an n×n ordinary chessboard. It is always possible to find n queens positioned in such a way, that no two attack each other. This is although only true, when $n \geq 4$. Another way to pose the question is: How many such placements can be found, when there are no such two queens who share a row, column or a diagonal?

The original problem was only for 8 queens on a regular chessboard. For the original 8 queens' problem, 92 solutions to this date have been identified. Of these 92, there are 12 distinct patterns. Thus all of the 92 solutions can be transformed into the 12 unique patterns, using reflection and rotation. These 12 patterns are show in Table 1.

Sol.	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6	Row 7	Row 8
1	1	5	8	3	3	7	2	4
2	1	6	8	7	7	4	2	5
3	2	4	6	3	3	1	7	5
4	2	5	7	3	3	8	6	4
5	2	5	7	1	1	8	6	3
6	2	6	1	4	4	8	3	5
7	2	6	8	1	1	4	7	5
8	2	7	3	8	8	5	1	4
9	2	7	5	8	1	4	6	3
10	3	5	2	8	1	7	4	6
11	3	5	8	4	1	7	2	6
12	3	6	2	5	8	1	7	4

Table 1: 12 unique patterns for the 8-queen problem.

For instance, if one is to constructing solution number 1, then the queen for chessboard row 1 should be placed in column 1, the queen for row 2 should be placed in column 5, and so on.

Modular Chessboard N-queen problem. A modular chess board is a one where the diagonals run on the other side of the board. Thus a queen can still be under attack, even if it is not directly under attack from another queen on the diagonal. This fact is illustrated in figure 1. The basic question asked for a modular chessboard is: What is the maximum number of queens that can be accommodated on a modular chessboard, such that no two Queens attack each other? In other words if an $n \times n$ chessboard is transformed into a torus by identifying the opposite side, we want to place n queens in such a way that none attack each other. This number is denoted as M(n). The basic modular chessboard n-queen problem is accredited to Polya [Polya 1921].

Known Results. There has been a lot of work done on the modular n-queen problem. The first major result was by Polya [Polya 1921]. He proved that if and only if gcd(n,6)=1 then there are n queens on the modular board. The same result was proved by many others see

[Heden 1992]. Later Klöve [Klöve 1981], improved the result and gave the following theorem:

Theorem (Klöve)**.** The modular chessboard has solutions of the following form:

i. M(n)=n if gcd(n,6)=1,
ii. M(n)=n-2 if gcd(n,6)=3,
iii. n-3 ≤ M(n) ≤ n-1 if gcd(n,6)=2,
iv. n-5 ≤ M(n) ≤ n-1 if gcd(n,6)=6,

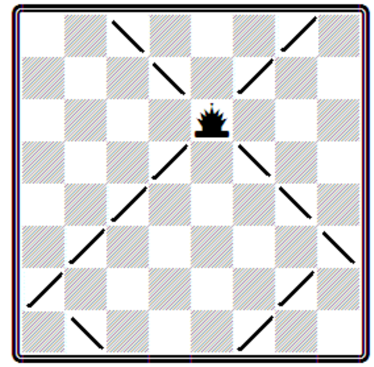

Figure 1: Modular chessboard

Later Heden [Heden 1992] further improved Klöve's work and gave a much simpler proof and summarized the results as follows:

Theorem (Heden)**.** The modular chessboard has solutions of the following form:

i. M(n)=n if gcd(n,6)=1,
ii. M(n)=n-1 if gcd(n,12)=2,
iii. M(n)=n-2 if gcd(n,6)=3 or 4,
iv. n-4 ≤ M(n) ≤ n-2 if gcd(n,12)=6,
v. n-5 ≤ M(n) ≤ n-2 if gcd(n,12)=12,

Heden used the concept of *chains of queens* and *colored queens* to prove his results. A chain of queens on a diagonal is a set of queens (Q_1, Q_2, Q_3,…. Q_k) such that the following two conditions hold:

i. No three queens are on the same diagonal or bi-diagonal.
ii. Two queens marked consecutively are always in the same diagonal or bi-diagonal.

A chain is closed, if Q_1 = Q_k. Similarly, chaining can be defined for rows and columns. Heden defined four colors for queens as A, B, C and D. A queen colored in color A is called A-queen. A queen colored in either A or D is called AD-queen, and so on. This helped in defining the colorings for the diagonal and bi-diagonal, which are the bases of his proof.

This approach is so much so effective, that he was able to give a partial solution without the aid of computer towards the modular 12-double queen problem, with 22 queens. A *modular double queen problem* is an extension of the normal modular queen problem, where there can be at most two queens on a single row, column, or a diagonal.

Open Problems.

i. If M(n) denotes the maximum number of queens on a modular chessboard such that no two queens attack each other, then is there a proof for M(n) = n-2 if 6 divides n?

ii. For higher dimensional play of modular n-queen problem, Nudelman [Nudelman 1995] concluded that if there exists a complete solution for d-dimensional modular n-queen problem, than no prime less than 2d can divide n. Can someone determine M(n,d) (where n is the number of queens and d is the dimension) in case of incomplete solutions?

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