Some Preliminary Results of Three Combinatorial Board Games

Samee Ullah Khan* Ishfaq Ahmad†

Abstract

This paper analyzes a certain class of combinatorial board games that includes Ayo, Tchoukaillon and Modular *N*–queen. We refine the existing results by providing simple and intuitive proofs using abstract combinatorics, and also reveal some interesting game positions.

1 Introduction

Chess and its variants, are pure strategic games without any random moves. It should, at least in principle, be possible to decide whether there is a winning strategy for the first or the second player, or whether the game always ends with a draw (assuming a perfect play). Such games are called combinatorial games, and combinatorial game theory (CGT) is a branch of mathematics devoted to their analysis [2]. A rich theory on how to evaluate game positions has been developed in recent years, and it has been successfully applied to analyze certain endgame positions of Chess [6] and Go [1], for example. Unfortunately, CGT cannot directly be applied to Chess, Shogi and Xiangqi, due to the fact that draws do not qualify as a game in CGT [2, 4]. Moreover, CGT also fails to help analyze various other board games such as *Ayo*, *Tchoukaillon* and *Modular N–queen* which are classified as pure combinatorial games [7, 9, 12]. In this paper we analyze these three board games with the help of abstract combinatorics and report some preliminary results.

2 Ayo

There are various kinds of Mancala games that date back to the early years of the great Egyptian Empire [10]. Mancala games mostly played in Nigeria are a group

*Department of Computer Science and Engineering, University of Texas at Arlington, sakhan@cse.uta.edu. †Department of Computer Science and Engineering, University of Texas at Arlington, iahmad@cse.uta.edu.

of games, with certain common characteristics. They all involve cup–shaded depressions called pits that are filled with stones or seeds. Players take turns and maneuver the stones, by various rules, which govern them. Ayo is one such two–player game played mostly in the western part of Nigeria. With predefined rules, Ayo players follow their respective strategies and do not depend on chance moves (dice).

Ayoyayo (Ayo) is played over a wooden board 20 inches long, 8 inches wide, and 2 inches thick. This board accommodates two rows of six pits each 3 inches in diameter. The pits are filled with either stones or dried palm nuts [3].

Ayo is played with 48 stones with 4 stones placed in each of the 12 pits. Two players alternatively move the stones, each controlling 6 pits. Their objective is to capture their opponent's stones (as many as possible). A move consists of a player choosing a non–empty pit on his side of the board and removing all of the stones contained in that pit. The stones are redistributed (*sown*) one stone per pit. The pits are sown in counter–clockwise direction from the pit that has been just emptied. A pit containing 12 or more stones is called an *Odu*. If the chosen pit is an *Odu*, the same redistribution continues, but the emptied pit is skipped on each circuit of the board [10]. A capture is made, when the last pit sown is on the opponent's side, and contains after the addition of the sowing stone either two or three stones. Thus, the stones in the pit are captured and removed from the game. Also are captured the immediately preceding pits which meet the same conditions. One important feature of this game is that each player has to make a move such that his opponent has a legal move to play. If this does not happen, then the opponent is rewarded with all the remaining stones on the board. If during the game, it is found that there are not enough stones to make a capture, but both the players can always proceed with a legal move, the game is stopped and the players are awarded stones that reside on their respective side of the board. The initial game is rapid and much more interesting, where both the players capture stones in quick succession. To determine the optimal strategy during the initial play is hard, and thus has not yet been studied. It involves planning at least 2–3 moves in advance, and remembering the number of stones in every pit [3, 10].

To the best of the authors' knowledge, the only work reported on Ayo can be found in [3]. They generalized Ayo such that: 1) Each player got n pits, 2) the *Odu* rule was overridden, and 3) the Ayo board was numbered in clockwise $-n+2, -n+1, \cdots, -1, 0, 1, \cdots, n, n+1$. Their analysis was only confined to the Ayo endgames. They defined a determinable position as an arrangement of stones where it is possible for a player to move such that:

1. A player captures at every turn.

2. No move is allowed from *Odu*.

3. After a player has moved, his opponent has only one stone on his side of the board.

4. Every stone is captured except the one which is award to his opponent.

Based on these assumptions, Broline and Loeb, also showed a small endgame with nine stones. Thus they provided us with the following lemma.

Lemma 1. *If a player has to move in a determined Ayo position, his stone has to be in pit* 1*, else in pit* 0 *if his opponent has to move.*

The proof presented in [3] is very complicated and tedious. Here we present a simple and intuitive proof (the pit positions are illustrated in Figure 1).

Proof: From the predefined set of four legal moves, if a player's opponent has to make the move, he must capture (step 1) and leave only one stone (step 3). With simple combinatorics it can be shown that the stone has to be in pit 0 (step 4). Thus before the player's move, the stone has to be in pit 1.

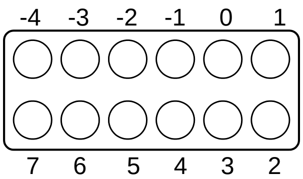

Figure 1: Ayo board labelled with pit numbers.

3 Tchoukaillon

Tchoukaillon is a Russian game, played with pits dug into sand and the pits filled with stones. The game is the modern variation of *Tchouka* which was mostly played in central Europe. In [5] the authors developed Tchoukaillon as the modern variant of Tchouka. This game also falls in the category of Mancala games [9]. The modern version of Tchoukaillon involves playing over a strip of wood (also possible is the circular arrangement of the pits). These pits contain a certain number of stones, with one empty pit called *Rouma*, *Cala* or *Roumba*.

There is no limit as to how many stones can to be used, and neither is a limit on the number of pits [12]. The objective of the game is to put the stones in *Roumba*. The maneuvering of the stones is called *sow*. Thus, like a solitaire game stones are sown into the empty pit. The sowing takes place as a constant one stone per pit at a time in the direction of *Roumba*, but it can also be in the opposite direction of *Roumba*. Therefore, there can be only three possibilities during the game:

1. If the last stone drops into *Roumba*, the player has a choice to start sowing another pit of his choice.
2. If the last stone drops in an occupied non-*Roumba* pit, this pit is to be sown immediately.
3. If the last pit drops in an empty non-*Roumba* pit, the game is over and the player who does this losses.

The objective in a two–player Tchoukaillon game is to play the last stone in an empty pit so that the next player takes the turn. While doing so he has to sow as many stones in *Roumba* as possible. The winner of the game is the player with the largest number of stones in *Roumba*. The Tchoukaillon board is shown in Figure 2.

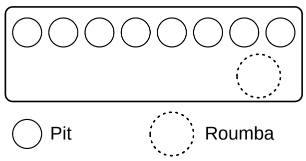

Figure 2: Tchoukaillon board.

To the best of the authors' knowledge, the work reported in [9] and [3] are the only ones that address Tchoukaillon as a combinatorial game. In [3] the authors investigated some winning positions, but disregarded a number of other possibilities of winning and confined themselves to just the objective of sowing stones in *Roumba* only. Their strategy is based on the results based on the work of [5]. They stated the winning move as:

If a win is possible from a given Tchoukaillon position, the unique winning move must be to harvest the smallest harvestable pit.

Although the authors in [3] identified a winning strategy, yet they did not provide any proof of the strategy. Here we state a simple proof.

Lemma 2. *The unique winning move in Tchoukaillon is to harvest the smallest harvestable pit.*
Proof: Imagine if there are only two pits on the board, with one pit having less

number of stones than the other. If a stone is taken from the pit having more stones, it will increase the number of stones in the smaller pit. Thus the pit at some moment will overflow if a play is continued in this fashion, and an indefinite play will continue.

From Lemma 1, one can always argue that there should be a way to backtrack all possible positions till the initial setup. Based on this argument one can always make the following claim.

Conjecture 1. *For all $s = 0$ (initial setup of the game), there is exactly one winning position involving a total of s stones.*

4 Modular N–queen

Consider an $n \times n$ ordinary chessboard. It is always possible to find n queens positioned in such a way, that no two attack each other. This is although only true, when $n = 4$. There are a number of ways to pose the modular n–queen problem, e.g., How many such placements can be found, when there are no such two queens who share a row, column or a diagonal? The original problem was only for 8 queens on a regular chessboard. For the original 8 queens' problem, 92 solutions to this date have been identified. Of these 92, there are 12 distinct patterns. Thus all of the 92 solutions can be transformed into the 12 unique patterns, using reflection and rotation. These 12 patterns are show in Table 1. For instance, if one is to constructing solution number 1, then the queen for chessboard row 1 should be placed in column 1, the queen for row 2 should be placed in column 5, and so on.

A modular chess board is a one where the diagonals run on the other side of the board. Thus a queen can still be under attack, even if it is not directly under attack from another queen on the diagonal. This fact is illustrated in Figure 3. The basic question asked for a modular chessboard is: What is the maximum number of queens that can be accommodated on a modular chessboard, such that no two queens attack each other? In other words if an $n \times n$ chessboard is transformed into a torus by identifying the opposite side, we want to place n queens in such a way that none attack each other. This number is denoted as $M(n)$. The basic modular chessboard n–queen problem is accredited to Pölya [11].

There has been a lot of work done on the modular n–queen problem. The first major result was reported in [11]. There the author proved that if and only if $gcd(n, 6) = 1$ then there are n queens on the modular board. The same result was proved by many others, e.g. see [7]. Later Klöve improved the result and gave the following theorem [8].

Theorem 1. *The modular chessboard has solutions of the following form:*

Solution	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6	Row 7	Row 8
1	1	5	8	3	3	7	2	4
2	1	6	8	7	7	4	2	5
3	2	4	6	3	3	1	7	5
4	2	5	7	3	3	8	6	4
5	2	5	7	1	1	8	6	3
6	2	6	1	4	4	8	3	5
7	2	6	8	1	1	4	7	5
8	2	7	3	8	8	5	1	4
9	2	7	5	8	1	4	6	3
10	3	5	2	8	1	7	4	6
11	3	5	8	4	1	7	2	6
12	3	6	2	5	8	1	7	4

Table 1: 12 unique patterns for the 8–queen problem.

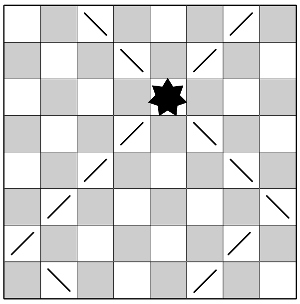

Figure 3: Modular chessboard.

1. $M(n) = n$ *if* $gcd(n, 6) = 1$
2. $M(n) = n - 2$ *if* $gcd(n, 6) = 3$
3. $n - 3 = M(n) = n - 1$ *if* $gcd(n, 6) = 2$
4. $n - 5 = M(n) = n - 1$ *if* $gcd(n, 6) = 6$

Proof: [8]

Later, Heden further improved Klöve's result by providing a simpler proof of the original work reported in [11]. The following theorem is deduced from the work reported in [7].

Theorem 2. *The modular chessboard has solutions of the following form:*

1. $M(n) = n$ *if* $gcd(n, 6) = 1$

2. $M(n) = n - 1$ *if* $gcd(n, 12) = 2$

3. $M(n) = n - 2$ *if* $gcd(n, 6) = 3$ *or* 4

4. $n - 4 = M(n) = n - 2$ *if* $gcd(n, 12) = 6$

5. $n - 5 = M(n) = n - 2$ *if* $gcd(n, 12) = 12$

Proof: [7]

Heden used the concept of chains of queens and colored queens to prove his results. A chain is closed, if $Q_1 = Q_k$. Similarly, chaining can be defined for rows and columns. Heden defined four colors for queens as A, B, C and D. A queen colored in color A is called A–queen. A queen colored in either A or D is called AD–queen, and so on. This helped in defining the colorings for the diagonal and bi–diagonal, which are the bases of his proof. This approach is so effective, that he was able to give a partial solution without the aid of computer towards the modular 12–double queen problem, with 22 queens. A modular double queen problem is an extension of the normal modular queen problem, where there can be at most two queens on a single row, column, or a diagonal.

Here, we generalize the chains of queens concepts and give the following necessary and sufficient conditions.

Conjecture 2. *A chain of queens on a diagonal is a set of queens (Q_1, Q_2, Q_3, $\cdots$, Q_k) such that the following two conditions hold:*

1. *No three queens are on the same diagonal or bi–diagonal.*

2. *Two queens marked consecutively are always in the same diagonal or bi–diagonal.*

5 Concluding Remarks and Some Open Problems

In this paper we provided the readers with some preliminary results on a class of combinatorial board games that includes Ayo, Tchoukaillon and Modular N–queen. These games require abstract combinatorial analysis and cannot be analyzed by pure combinatorial game theoretical methods. Thus, the combinatorial game theory lacks some fine tuning especially in case of combinatorial board games (loopy games). We now pose the following open problems:

Problem 1: Is it possible to identify a winning sequence of moves provided an arbitrary board position in Tchoukaillon?

We conjecture that this is possible, but it is to be noted that the game played with n number of pits might involve enormous amount of moves (Conjecture 1). It is thus, *also* possible that the problem might not be intractable to begin with.

Problem 2: Are there more than two necessary and sufficient conditions to form a chain of queens on a diagonal?

References

[1] E. R. Berlekamp and D. Wolfe. *Mathematical Go–Chilling Gets the Last Point*. A K Peters, Natick, MA, 1994.

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for your Mathematical Plays*, volume I & II. Academic Press, London, 1982. 2nd edition of vol. 1 (of four volumes), 2001, A. K. Peters, Natick, MA.

[3] D. M. Brolin and D. E. Loeb. The combinatorics of mancala-type games. *UMAP*, 10(1), 1995.

[4] J. H. Conway. *On Numbers and Games*. Academic Press, London, 1976.

[5] A. Deledicq and A. Popova. Wari et solo: Le jeu de calculs africain. *Collection Les Distracts*, 3:318–359, 1977.

[6] N. D. Elkies. On numbers and endgames: combinatorial game theory in chess endgames. In R. J. Nowakowski, editor, *Games of No Chance*, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA, MSRI Publ., volume 29, pages 135–150. Cambridge University Press, Cambridge, 1996.

[7] O. Heden. On the modular n–queen problem. *Discrete Mathematics*, 102:155–161, 1992.

[8] T. Klöve. The modular n–queen problem II. *Discrete Mathematics*, 1981:33–48, 1981.

[9] D. E. Loeb. Combinatorial properties of mancala. *Abstratcs of AMS*, 96:471, 1994.

[10] A. O. Odeleya. *Ayo: A popular Yoruba Game*. Oxford University Press, Ibadan, Nigeria, 1977.

[11] G. Polya. Uber die “doppelt–periodishen” lösungen des n–damen–problem. In W. Ahrens, editor, *Mathematische Unterhaltungen and Spiele*, volume 2. 1921.

[12] M. A. Saint-Laguë. Géométrie de situation et jeux. *Mémorial des Sciences Mathématiques*, Fascicule XLI, 1929.